Elevator Pitch

- **featured transition systems** for modeling software product lines
  - transitions can be turned on and off depending on available features

- **weighted automata** for modeling quantitative systems
  - shortest path; maximum flow; energy consumption; probabilities

Here: **featured weighted automata** for quantitative properties of SPLs

Key lemma: A featured weighted automaton is a weighted automaton
1. Motivation

2. Finite Runs in Semiring-Weighted Automata

3. Featured Weighted Automata

4. Conclusion
Motivation

A weighted automaton is shown with transitions labeled with operations such as `cancel [C]`, `dep [D]`, `more [M ∧ R]`, and `more [M ∧ ¬R]`. The states are labeled with events like `card?`, `PIN`, `amount`, `cash`, `rec [R]`, and `card! [¬R]`.

The automaton transitions are as follows:
- From state 0 to state 1: `card?`
- From state 1 to state 2: `PIN`, `card?`
- From state 2 to state 3: `amount`, `cancel [C]`, `more [M ∧ R]`
- From state 3 to state 4: `cash`, `dep [D]`, `more [M ∧ ¬R]`, `cancel [C]`
- From state 4 to state 5: `rec [R]`, `card! [¬R]`, `card! [R]`
Motivation
Motivation

Motivation Finite Runs in Semiring-Weighted Automata

Featured Weighted Automata

Conclusion

Uli Fahrenberg Axel Legay

Featured Weighted Automata

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Motivation
Motivation

Finite Runs in Semiring-Weighted Automata

Featured Weighted Automata

Conclusion

Motivation
Motivation

Quantitative analysis of FTS:
- Cordy, Schobbens, Heymans, Legay ICSE 2013: Beyond Boolean product-line model checking

Here: Generalization to semiring-weighted FTS
- semiring-weighted automata for modeling and analysis of different types of quantitative properties
- Kleene algebra
- key lemma: An FTS on products $px$ weighted in a semiring $K$ is an automaton weighted in the function semiring $px \rightarrow K$
Semirings

A semiring is a structure \((K, \oplus, \otimes, 0, 1)\) such that

- \((K, \oplus, 0)\) is a commutative monoid,
  
  \[ x \oplus y = y \oplus x, \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad x \oplus 0 = x \]

- \((K, \otimes, 1)\) is a monoid,
  
  \[ x \otimes (y \otimes z) = (x \otimes y) \otimes z, \quad x \otimes 1 = 1 \otimes x = x \]

and which satisfies distributive and annihilation laws:

- \[ x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z, \quad (x \oplus y) \otimes z = x \otimes z \oplus y \otimes z \]
- \[ x \otimes 0 = 0 \otimes x = 0 \]

Examples:

- natural numbers: \((\mathbb{N}, +, \cdot, 0, 1)\)
- languages over some alphabet \(\Sigma\): \((2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})\)
- Boolean semiring: \((\{\text{ff, tt}\}, \lor, \land, \text{ff}, \text{tt})\)
Semiring-Weighted Automata

Let $K$ be a semiring. A $K$-weighted automaton is a tuple $S = (S, I, F, T)$:

- $S$ finite set of states, $I \subseteq S$ initial, $F \subseteq S$ accepting
- $T \subseteq S \times K \times S$ finite set of transitions

An accepting path in $S$: finite sequence $\pi = (s_0, x_0, s_1, \ldots, x_k, s_{k+1})$ of transitions $(s_0, x_0, s_1), \ldots, (s_k, x_k, s_{k+1}) \in T$, with $s_0 \in I$ and $s_{k+1} \in F$

- weight of $\pi$: $w(\pi) = x_0 \otimes \cdots \otimes x_k$

The reachability value of $S$:

$$|S| = \bigoplus \{w(\pi) \mid \pi \text{ accepting path in } S\}$$

- if this infinite sum exists in $K$
Semiring-Weighted Automata: Examples

Recall
- for $\pi = (s_0, x_0, s_1, \ldots, x_k, s_{k+1})$: $w(\pi) = x_0 \otimes \cdots \otimes x_k$
- $|S| = \bigoplus \{w(\pi) \mid \pi$ accepting path in $S\}$

**Boolean semiring** ($\{\ttt, \ff\}, \lor, \land, \ttt, \ff$):
- $w(\pi) = \ttt$ iff all $x_i = \ttt$
- $|S| = \ttt$ iff an accepting state is reachable (through $\ttt$-labeled transitions)

**Tropical semiring** ($\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0$):
- $w(\pi) = x_0 + \cdots + x_k$
- $|S| = \text{minimum reachability value / shortest path}$

**Fuzzy semiring** ($\mathbb{R}_{\geq 0} \cup \{\infty\}, \max, \min, 0, \infty$):
- $w(\pi) = \min\{x_0, \ldots, x_k\}$
- $|S| = \text{maximum flow}$
Conway Semirings

A star semiring is a semiring \((K, \oplus, \otimes, 0, 1)\) with a star operation
\[ * : K \rightarrow K \]
- intuition: \(\oplus\) for choice, \(\otimes\) for composition, \(*\) for iteration

A Conway semiring is a star semiring \((K, \oplus, \otimes, *, 0, 1)\) in which

\[
(x \otimes y)^* = 1 \oplus x \otimes (y \otimes x)^* \otimes y \\
(x \oplus y)^* = (x^* \otimes y)^* \otimes x^*
\]

- encodes properties of iteration

Examples:
- Boolean: \(x^* = \tt\)
- Tropical: \(x^* = 0\)
- Fuzzy: \(x^* = \infty\)
Matrix Semirings

Let $K$ be a semiring and $n \geq 1$. The **matrix semiring** over $K$ is $(K^{n \times n}, \oplus, \otimes, 0, 1)$

- standard matrix addition and multiplication, like in linear algebra; $0 =$ zero matrix, $1 =$ identity matrix

Old theorem: If $K$ is a Conway semiring, then so is $K^{n \times n}$

- with $M_{i,j}^* = \bigoplus_{m \geq 0} \bigoplus_{1 \leq k_1, \ldots, k_m \leq n} M_{i,k_1} \otimes M_{k_1,k_2} \otimes \cdots \otimes M_{k_m,j}$

- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^* = \begin{bmatrix} (a \oplus b \otimes d^* \otimes c)^* & (a \oplus b \otimes d^* \otimes c)^* \otimes b \otimes d^* \\ (d \oplus c \otimes a^* \otimes b)^* \otimes c \otimes a^* & (d \oplus c \otimes a^* \otimes b)^* \end{bmatrix}$$

(recursively)
Matrix Semirings

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(recursively)
Automata Weighted in Conway Semirings

Let $K$ be a Conway semiring

- a $K$-weighted automaton (with $n$ states): $S = (\alpha, M, \kappa)$
- $\alpha \in \{0, 1\}^n$ initial vector, $\kappa \in \{0, 1\}^n$ accepting vector,
  $M \in S^{n \times n}$ transition matrix
- (equivalent to representation $S = (S, I, F, T)$ )
- Recall $|S| = \bigoplus \{w(\pi) | \pi$ accepting path in $S\}$
  (if it exists in $K$)
- Old theorem: $|S|$ exists in $K$, and $|S| = \alpha M^* \kappa$
Featured Weighted Automata

Let $K$ be a Conway semiring and $px$ a set of products. A featured $K$-weighted automaton is a tuple $F = (S, I, F, T)$:

- $S$ finite set of states, $I \subseteq S$ initial, $F \subseteq S$ accepting
- $T \subseteq S \times [px \to K] \times S$ finite set of transitions

Let $p \in px$. The projection of $F$ to $p$ is the $K$-weighted automaton $\text{proj}_p(F) = (S, I, F, T')$, with $T' = \{(s, f(p), s') \mid (s, f, s') \in T\}$

- We are interested in the values $|\text{proj}_p(F)|$ for all $p \in px$

**Theorem:** $[px \to K]$ is a Conway semiring, with

$(f \oplus g)(p) = f(p) \oplus g(p)$, $(f \otimes g)(p) = f(p) \otimes g(p)$, and $f^*(p) = (f(p))^*$, and for all $p \in px$,

$$|\text{proj}_p(F)| = |F|(p)$$

- family-based analysis: compute $|F|$
Featured Weighted Automata, Symbolically

Recall: A featured $K$-weighted automaton is a tuple $\mathcal{F} = (S, I, F, T)$ with $T \subseteq S \times [px \rightarrow K] \times S$

- for computations in semiring $[px \rightarrow K]$, need good symbolic representation of functions $px \rightarrow K$

Let $N$ be a set of features, so that $px \subseteq 2^N$.

- $\mathbb{B}(N)$: Boolean expressions over $N$ (feature guards)
- for $\gamma \in \mathbb{B}(N)$: $\llbracket \gamma \rrbracket =$ all products which satisfy $\gamma$
- guard partition: $P \subseteq \mathbb{B}(N)$ such that $\llbracket \bigvee P \rrbracket = px$, $\forall \gamma \in P : \llbracket \gamma \rrbracket \neq \emptyset$, and $\forall \gamma_1 \neq \gamma_2 \in P : \llbracket \gamma_1 \rrbracket \cap \llbracket \gamma_2 \rrbracket = \emptyset$

Let $GP[K] = \{ f : P \rightarrow K \mid P$ guard partition, $\forall \gamma_1 \neq \gamma_2 \in P : f(\gamma_1) \neq f(\gamma_2) \}$

- injective functions from guard partitions to $K$
(f ⊕ g)(p) = f(p) ⊕ g(p)

1: function $K\text{SUM}(f_1 : P_1 \rightarrow K, f_2 : P_2 \rightarrow K): GP[K]$
2: var $f', P'$
3: $P' \leftarrow \emptyset$
4: for all $\gamma_1 \in P_1$ do
5:   for all $\gamma_2 \in P_2$ do
6:     if $[\gamma_1 \land \gamma_2] \neq \emptyset$ then
7:       $P' \leftarrow P' \cup \{\gamma_1 \land \gamma_2\}$
8:       $f'(\gamma_1 \land \gamma_2) \leftarrow f_1(\gamma_1) \oplus f_2(\gamma_2)$
9: return $K\text{COMBINE}(f')$
\[(f \otimes g)(p) = f(p) \otimes g(p)\]

1: \textbf{function} $\text{KProd}(f_1 : P_1 \to K, f_2 : P_2 \to K)$: \text{GP}[K]

2: \textbf{var} $f'$, $P'$

3: $P' \leftarrow \emptyset$

4: \textbf{for all} $\gamma_1 \in P_1$ \textbf{do}

5: \hspace{1em} \textbf{for all} $\gamma_2 \in P_2$ \textbf{do}

6: \hspace{2em} \textbf{if} $\lfloor \gamma_1 \land \gamma_2 \rfloor \neq \emptyset$ \textbf{then}

7: \hspace{3em} $P' \leftarrow P' \cup \{\gamma_1 \land \gamma_2\}$

8: \hspace{3em} $f'(\gamma_1 \land \gamma_2) \leftarrow f_1(\gamma_1) \otimes f_2(\gamma_2)$

9: \textbf{return} $\text{KCombine}(f')$
Featured Weighted Automata, Computationally

\[ f^*(p) = (f(p))^* \]

1. function \( K\text{STAR}(f : P \rightarrow K) : GP[K] \)
2. \hspace{1em} var \( f' \)
3. \hspace{1em} for all \( \gamma \in P \) do
4. \hspace{2em} \( f'(\gamma) \leftarrow f(\gamma)^* \)
5. \hspace{1em} return \( K\text{COMBINE}(f') \)
Featured Weighted Automata, Computationally

1: function $K \text{COMBINE}(f : P \to K): GP[K]$

2: var $\tilde{f}$, $\tilde{P}$

3: $\tilde{P} \leftarrow \emptyset$

4: while $P \neq \emptyset$ do

5: Pick and remove $\gamma$ from $P$

6: $x \leftarrow f(\gamma)$

7: for all $\delta \in P$ do

8: if $f(\delta) = x$ then

9: $\gamma \leftarrow \gamma \lor \delta$

10: $P \leftarrow P \setminus \{\delta\}$

11: $\tilde{P} \leftarrow \tilde{P} \cup \{\gamma\}$

12: $\tilde{f}(\gamma) \leftarrow x$

13: return $\tilde{f} : \tilde{P} \to K$
family-based analysis of featured weighted automata is easy in theory
  because featured weighted automata are weighted automata
in practice:
  Cordy, Schobbens, Heymans, Legay ICSE 2013: featured shortest paths
  Olaechea, U.F., Atlee, Legay SPLC 2016: featured long-term average
both show that family-based analysis is better than product-based
  but not always, and not much
  problem: partition splitting
The purpose of DHS is to connect people working in real-time systems, hybrid systems, control theory, distributed computing, and concurrency, in order to advance the subject of distributed hybrid systems.

Distributed hybrid systems, or distributed cyber-physical systems, are abundant, but ensuring their correct functioning is very difficult. We believe that convergence and interaction of methods and tools from different areas of computer science, engineering, and mathematics is needed in order to advance the subject.

This first edition of the DHS workshop aims at gathering researchers which work in the above areas in order to facilitate collaboration and discuss how the subject may advance.

Invited Speakers

Martin Fränzle  
Oldenburg, Germany

Kim G. Larsen  
Aalborg, Denmark

Sergio Rajsbaum  
Mexico City, Mexico

Martin Raussen  
Aalborg, Denmark

Rafael Wisniewski  
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